Indexed Virtual Number Algebra: A Structured Interface for Division by Zero and Indeterminate Forms

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"The limit does not exist!"
— Cady Heron, *Mean Girls* (2004)

Abstract

We present **Indexed Virtual Number Algebra (IVNA)**, an algebraic framework that attaches real-valued indices to zeros and infinities, making division by zero and indeterminate forms algebraically operable. Where standard mathematics returns "undefined" for expressions like 5/0, IVNA returns ∞_5—an indexed infinity that preserves the numerator's identity and permits continued computation, including exact recovery of the original value via $\infty_5 \cdot 0_1 = 5$. More generally, $y/0_x = \infty_{\frac{y}{x}}$ for any real y and index x.

The key innovation is the **indexed product rule**: $0_x \cdot \infty_y = xy$, which resolves the product of zero and infinity to a determinate finite number by tracking provenance

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Methodology: The core conceptual framework—indexed zeros and infinities, the product rule $0_x \cdot \infty_y = xy$, and the applications to calculus and indeterminate forms—was conceived and developed by the author beginning in 2017, well before any AI tooling was involved. The formalization, computational verification, and paper preparation were then carried out using the Playful Sincerity Digital Core—an AI-assisted research methodology system built on Claude Code (Anthropic). The Digital Core enforces a Generate–Verify–Revise (GVR) loop—inspired by the Aletheia architecture [21]—in which every mathematical claim is verified by at least one independent computational tool and revised if verification fails. The system orchestrates hierarchical planning, parallel agent-based exploration (literature search, symbolic verification, and formal proof running concurrently), and book/paper research integration. Six independent verification tool chains—Python, SymPy, Z3, Lean 4, Wolfram, and a meta-verification layer—produced 403 automated checks with zero failures, each honestly categorized by what it tests (see Appendix A). The complete verification suite is publicly available and independently reproducible at `https://github.com/Playful-Sincerity/PS-Research-IVNA`.

through indices. This rule has no exact precedent in Non-Standard Analysis, grossone, wheel algebra, surreal numbers, or smooth infinitesimal analysis.

We prove IVNA consistent by constructing an explicit model via embedding in Robinson's hyperreal numbers, where $0_x = x\varepsilon_0$ for a fixed infinitesimal ε_0. This embedding is verified computationally (70 symbolic checks via SymPy, 13 satisfiability and independence checks via Z3, 42 cross-checks via Wolfram) and formally (core axioms type-checked in Lean 4 with zero proof gaps). IVNA is thus a structured notational interface to a specific fragment of Non-Standard Analysis—analogous to how complex number notation ($a+bi$) is an interface to $\mathbb{R}^2$ with a specific multiplication rule.

We demonstrate applications in limit-free calculus (derivatives of polynomial, rational, trigonometric, exponential, and logarithmic functions without ε-δ arguments), elimination of L'Hôpital's rule, proportional infinite set sizes, and a "VEA mode" for computer arithmetic that resolves IEEE 754 NaN propagation when operand indices are tracked. The exponential constant e receives a direct algebraic definition: $e = (1 + 0_1)^{\infty_1}$, revealing the structure of continuous growth as a scaling symmetry between step size and repetition count. Beyond calculus, the product rule $0_x \cdot \infty_y = xy$ recurs as the resolution mechanism in distribution theory (the Dirac delta), probability (conditional densities and Bayes' theorem), and algebraic geometry (a new correspondence between IVNA's index arithmetic and blow-up resolution)—a structural observation suggesting that these independently developed techniques share a common algebraic core.

1 Introduction

1.1 The Problem

Consider the function $f(x) = 1/x$. As x approaches zero from the right, $f(x)$ grows without bound. As x approaches zero from the left, $f(x)$ decreases without bound. Standard mathematics concludes that $\lim_{x\to 0} 1/x$ does not exist and that $1/0$ is undefined.

But notice what this conclusion discards. The right-sided and left-sided behaviors are perfectly well-defined individually—they simply disagree. The "undefined" verdict arises not because nothing is happening, but because the single symbol 0 cannot distinguish two different zeros: the one approached from the right and the one approached from the left.

This is not an isolated case. The expressions $0/0$, $0\cdot\infty$, $\infty-\infty$, and ∞/∞ are all classified as *indeterminate forms*—not because they have no answer, but because the answer depends on information that the notation has discarded.[1] When we write $\lim_{x\to 0} \sin(x)/x = 1$ and $\lim_{x\to 0} x^2/x = 0$, both expressions involve the form $0/0$, yet they yield different values. The problem is not with zero or infinity. The problem is that standard notation uses a single symbol for objects that carry different weights.

The ε-δ framework of Weierstrass, the hyperreal construction of Robinson [1], and the grossone methodology of Sergeyev [6] each address this problem at different levels of formality. Yet in none of these frameworks is division by zero *algebraically operable*—a computation

[1] "Undefined" is mathematics' way of saying the question contains more information than the notation can hold.

that produces a result you can continue calculating with and eventually recover the original value from.

In IEEE 754 floating-point arithmetic [18], the situation is worse: `5.0 / 0.0` returns `Inf`, and `Inf * 0.0` returns `NaN`—"Not a Number." The information that the infinity came from dividing 5 by zero is destroyed. Every subsequent operation inherits the `NaN`, silently corrupting the computation.

1.2 A Precedent: Complex Numbers

This situation has a historical parallel. Before the sixteenth century, $\sqrt{-1}$ was considered impossible—an operation without a legitimate result. Cardano (1545) encountered it in his cubic formula and called such quantities "as refined as they are useless" (as quoted in [3]).

Bombelli (1572) took a different approach. He defined consistent rules for manipulating $\sqrt{-1}$ without knowing what it "was." The rules worked: imaginary intermediate steps in the cubic formula cancelled out and produced correct real answers. Gauss (1831) gave the geometric interpretation—complex numbers as points in a plane—and the objects became indispensable.

The complex numbers did not introduce new mathematics in the foundational sense. The ordered pair (a, b) with the multiplication rule $(a, b)(c, d) = (ac - bd, ad + bc)$ is an operation on $\mathbb{R}^2$. What the notation $a + bi$ provided was an *interface*: a way to work with $\mathbb{R}^2$ that made the algebraic structure transparent and the computations natural. The notation was the contribution.

We propose that division by zero is at a similar stage. The underlying mathematics exists—Robinson's Non-Standard Analysis provides a rigorous framework for infinitesimals and infinities. What is missing is an interface that makes division by zero algebraically operable.

1.3 IVNA's Proposal

Indexed Virtual Number Algebra (IVNA) attaches a nonzero *index* to every zero and every infinity. Instead of a single zero, there is a family $\{0_x : x \in \mathbb{C} \setminus \{0\}\}$. Instead of a single infinity, there is a family $\{\infty_x : x \in \mathbb{C} \setminus \{0\}\}$. The index x tracks the *weight*—informally, "how much zero" or "how much infinity" the object represents.

The central rule is the **indexed product**:

$$0_x \cdot \infty_y = xy. \tag{1}$$

The product of an indexed zero and an indexed infinity is a determinate finite number—the product of their indices. This resolves the indeterminate form $0 \cdot \infty$ by preserving the information that standard notation discards.

Division by zero follows immediately:

$$\frac{y}{0_x} = \infty_{\frac{y}{x}}. \tag{2}$$

The result is an indexed infinity whose index records the numerator and denominator. Crucially, the operation is *reversible*:

$$\infty_{\frac{y}{x}} \cdot 0_x = \frac{y}{x} \cdot x = y.$$

Multiply the indexed infinity by the indexed zero and you recover the original value. No information is lost.

The $1/x$ problem from Section 1.1 resolves cleanly. Approaching zero from the right means $x = 0_1$ (positive index); approaching from the left means $x = 0_{-1}$ (negative index):

$$\frac{1}{0_1} = \infty_1, \qquad \frac{1}{0_{-1}} = \infty_{-1}.$$

These are *different indexed infinities*. The two-sided limit "does not exist" because the two sides produce different objects—and IVNA makes this distinction explicit rather than collapsing both to "undefined."

1.4 Summary of Results

The contributions of this paper are:

1. **A consistent algebraic framework** (Section 2) in which division by zero produces operable results. Consistency is proven via explicit embedding in Robinson's hyperreals (Section 4), verified computationally (403 checks across six tools with zero failures), and formalized in Lean 4 (11 axioms, 12 theorems).

2. **A direct algebraic definition of** e (Section 5): $e = (1 + 0_1)^{\infty_1}$, revealing a scaling symmetry in continuous growth.

3. **Limit-free single-variable calculus** (Section 5): derivatives and integrals computed by direct algebraic manipulation, without ε-δ arguments or the standard part function. The Fundamental Theorem of Calculus reduces to an algebraic identity.

4. **Elimination of L'Hôpital's rule** (Section 5): the standard indeterminate forms $0/0$, ∞/∞, $0 \cdot \infty$, and $\infty - \infty$ resolved directly by index arithmetic, with 1^∞ handled by the Exponential Axiom (A-EXP).

5. **Applications to physics and computer science** (Sections 7–8): singularity classification in general relativity, and a "VEA mode" for computer arithmetic that resolves NaN propagation when operand indices are tracked.

6. **Honest positioning** (Section 9): IVNA is a structured notational interface to a specific fragment of Non-Standard Analysis, analogous to how $a + bi$ is an interface to $\mathbb{R}^2$. The indexed product rule $0_x \cdot \infty_y = xy$ has no exact precedent in the reviewed literature, but the underlying mathematics is not new.

7. **Cross-domain structural observation** (Section 6): the product rule $0_x \cdot \infty_y = xy$ recurs as the resolution mechanism across calculus, distribution theory, probability, and algebraic geometry—suggesting that these independently developed techniques share a common algebraic core.

2 Core Algebra

Before proceeding, we note that every axiom and theorem in this section is machine-verified. The complete Lean 4 formalization, a Python test suite (30 core tests), and the full verification suite (403 checks across six tools) are available at:

`https://github.com/Playful-Sincerity/PS-Research-IVNA`

To verify independently: clone the repository and run `python3 verification/run_all.py` (requires Python 3.10+, SymPy, Z3, and optionally Wolfram Engine). This single command runs all verification categories, checks the Lean 4 build, and saves detailed per-tool output to `verification/_results/`. We invite skeptical readers to verify before reading further.

2.1 Virtual Numbers

We begin with an example. In standard arithmetic, the expression $5/0$ is undefined, and the expression $0 \cdot \infty$ is indeterminate. In IVNA:

$$\frac{5}{0_1} = \infty_5, \qquad \infty_5 \cdot 0_1 = 5, \qquad 0_3 \cdot \infty_7 = 21.$$

Division by zero produces an indexed infinity. Multiplying it back by the indexed zero recovers the original. The product of an indexed zero and an indexed infinity yields a finite number—the product of their indices. We now define the objects and rules that make this work.

Definition 2.1 (Virtual Numbers)**.** A **virtual number** is either an *indexed zero* 0_x or an *indexed infinity* ∞_x, where $x \in \mathbb{C} \setminus \{0\}$ is the **index**. We call x the *weight* of the virtual number.

A **non-virtual number** is any element of $\mathbb{R}$ (or $\mathbb{C}$). We write n for a non-virtual number.

The term "virtual" reflects the position that exact zeros and exact infinities do not occur in physical measurement—they are constructs of mathematical models. This is a naming choice, not a claim about foundations. The reader may substitute "indexed" throughout without loss of content.

Remark 2.2 (Why "virtual"). More nothing will fit into a bigger thing than it would a smaller thing, even though it is still nothing. A swimming pool drained of water contains more "nothing" than a thimble. The index captures this: 0_5 represents the same mathematical zero as 0_1 under the collapse operator ($0_5 \fallingdotseq 0 = 0_1 \fallingdotseq 0$), but it is a zero with five times the capacity. In the NSA embedding (Section 4), $0_5 = 5\varepsilon_0$ is literally five times larger than $0_1 = \varepsilon_0$—still infinitesimal, but proportionally larger. These zeros are not the absence of quantity but the presence of an infinitesimal one.

Remark 2.3. The index domain is $\mathbb{C} \setminus \{0\}$, not $\mathbb{C}$. The excluded case $x = 0$ is addressed in Axiom 11 below: when an operation produces index zero, the result exits the virtual system to the real number 0. This is analogous to $i - i = 0$ exiting the imaginary numbers. For the core algebra (Sections 2–5), real indices suffice; complex indices are needed for directional information (Section 3).

2.2 Multiplication

The multiplication rules define how virtual numbers interact with each other and with non-virtual numbers. We begin with the central rule.

Axiom 1 (Zero–Infinity Product)**.** $0_x \cdot \infty_y = xy$

This is the core of IVNA. The product of an indexed zero and an indexed infinity is the product of their indices—a finite, non-virtual number. By commutativity (formalized in the Lean 4 development), $\infty_y \cdot 0_x = xy$ as well. The indices carry the information that makes the otherwise indeterminate product $0 \cdot \infty$ determinate.

Axiom 2 (Zero–Zero Product)**.** $0_x \cdot 0_y = 0^2_{xy}$

Axiom 3 (Infinity–Infinity Product)**.** $\infty_x \cdot \infty_y = \infty^2_{xy}$

The product of two indexed zeros is a *higher-order* indexed zero 0^2_{xy}, and similarly for infinities. The superscript tracks the order—how many times the virtual number has been "deepened" by multiplication with its own kind. Higher-order virtual numbers interact with their duals by order cancellation: $0^2_x \cdot \infty_y = 0_{xy}$ (one order cancels), and $0^n_x \cdot \infty^n_y = xy$ (n orders cancel completely).

Axiom 4 (Scalar–Zero Product)**.** $n \cdot 0_x = 0_{nx}$

Axiom 5 (Scalar–Infinity Product)**.** $n \cdot \infty_x = \infty_{nx}$

2.3 Division

Division rules follow from multiplication by duality. Dividing a non-virtual number by an indexed zero produces an indexed infinity, and vice versa. Dividing two virtual numbers of the same kind produces a non-virtual number—the ratio of their indices.

Axiom 6 (Division by Zero)**.** $\dfrac{y}{0_x} = \infty_{\frac{y}{x}}$

This is the rule that makes division by zero well-defined. The result is an indexed infinity whose index records both the numerator y and the denominator's weight x. For example, $5/0_1 = \infty_5$ and $5/0_2 = \infty_{5/2}$.

Axiom 7 (Division by Infinity)**.** $\dfrac{y}{\infty_x} = 0_{\frac{y}{x}}$

Axiom 8 (Zero–Zero Quotient)**.** $\dfrac{0_x}{0_y} = \dfrac{x}{y}$

Axiom 9 (Infinity–Infinity Quotient)**.** $\dfrac{\infty_x}{\infty_y} = \dfrac{x}{y}$

The last two axioms resolve two of the classical indeterminate forms. The expression $0/0$ is indeterminate in standard mathematics because the single symbol 0 discards the index information. In IVNA, $0_x/0_y = x/y$—a determinate ratio. Similarly, $\infty_x/\infty_y = x/y$. The indeterminacy existed because the notation lacked structure, not because the mathematics lacked answers.

2.4 Addition and Subtraction

Virtual numbers of the same kind and order combine by adding indices. Virtual numbers of different kinds coexist without combining, analogous to real and imaginary parts in $\mathbb{C}$.

Axiom 10 (Virtual Addition)**.** $0_x + 0_y = 0_{x+y}, \quad \infty_x + \infty_y = \infty_{x+y}$

This is consistent with the scalar rule: $0_x + 0_y = 0_{x+y}$ agrees with $1 \cdot 0_x + 1 \cdot 0_y = 0_x + 0_y$.

Remark 2.4 (Mixed addition)*.* The sum of a finite number and a virtual number is defined by the embedding: $n + \infty_x = \infty_x$ and $n + 0_x = n$ (up to the collapse operator). A finite quantity is infinitesimal relative to any indexed infinity and infinite relative to any indexed zero. In the hyperreal model, this is the standard fact that $r + x\omega_0 \sim x\omega_0$ for finite r.

Axiom 11 (Index Zero Rule (D-INDEX-ZERO))**.** When an operation produces index 0, the result exits the virtual number system to the real number 0. The index domain is $\mathbb{C} \setminus \{0\}$.

For example, $0_3 + 0_{-3} = 0$ (not 0_0). This is analogous to $i + (-i) = 0$ exiting the imaginary numbers to the real line. The rule ensures that when opposing virtual numbers cancel, the result is the real number zero—not a virtual number with a degenerate index.

2.5 Powers and Roots

Axiom 12 (Virtual Powers)**.** $(0_x)^n = 0^n_{x^n}, \quad (\infty_x)^n = \infty^n_{x^n}$

Remark 2.5 (Notation: order vs. index)*.* The superscript in 0^n_x denotes the *order*—the number of infinitesimal factors—not an exponent applied to the index. Thus $(0_x)^2 = 0^2_{x^2}$: the order becomes 2 and the index becomes x^2. These are related (raising a virtual number to the nth power produces order n with index x^n) but conceptually distinct: the order tracks depth, the index tracks provenance.

Remark 2.6*.* Virtual Powers and D-INDEX-ZERO are structural rules that follow naturally from the algebraic framework. The 11 core axioms formalized in Lean 4 comprise the nine multiplication and division rules above, plus the two addition rules (one for indexed zeros, one for indexed infinities—stated as a single axiom here but formalized separately).

2.6 The Collapse Operator

Definition 2.7 (Collapse ($\fallingdotseq$))**.** The **collapse operator** $\fallingdotseq$ strips indices from virtual numbers, projecting them to their standard values:

$$0_x \fallingdotseq 0, \qquad \infty_x \fallingdotseq \infty$$

for all $x \in \mathbb{C} \setminus \{0\}$. This is analogous to the standard part function st() in Non-Standard Analysis.

2.7 Duality and Reciprocals

Proposition 2.8 (Zero–Infinity Duality)**.** *Indexed zeros and indexed infinities are reciprocals:*

$$\frac{1}{0_x} = \infty_{\frac{1}{x}}, \qquad \frac{1}{\infty_x} = 0_{\frac{1}{x}}$$

2.8 Index Domain

Remark 2.9. The index domain is $\mathbb{C} \setminus \{0\}$. The restriction $x \neq 0$ is necessary: 0_0 would represent "zero with no weight," which under the multiplication rule $0_0 \cdot \infty_1 = 0 \cdot 1 = 0$ yields a result categorically different from $0_1^2 \cdot \infty_1 = 0_1$. The D-INDEX-ZERO axiom resolves this by routing index-zero results back to $\mathbb{R}$. Complex indices encode directional information: $0_{e^{i\theta}}$ represents a zero with approach angle θ.

Virtual-valued indices (e.g., 0_{0_z}) are permitted but reduce to order changes under the NSA embedding: $0_{0_z} = z\varepsilon_0^2 = 0_z^2$ (a higher-order zero), while $0_{\infty_z} = z$ exits to a real number. No recursive regress arises—virtual indices are syntactic sugar for the existing order hierarchy.

3 Extended Axioms

3.1 The Virtual Taylor Axiom

Axiom 13 (Virtual Taylor (A-VT))**.** For any function f analytic at a:

$$f(a + 0_x) = f(a) + 0_{f'(a)\cdot x} + 0^2_{\frac{f''(a)\cdot x^2}{2!}} + 0^3_{\frac{f'''(a)\cdot x^3}{3!}} + \cdots$$

Each term carries virtual order k and index $\frac{f^{(k)}(a)\cdot x^k}{k!}$. Under the collapse operator, all virtual terms vanish: $f(a + 0_x) \fallingdotseq f(a)$.

3.2 The Exponential Axiom

Axiom 14 (Exponential (A-EXP))**.** $(1 + 0_x)^{\infty_y} = e^{xy}$

Remark 3.1. A-EXP is a corollary of A-VT applied to $f(h) = (1 + h)^{1/h}$ in a suitable sense, but we state it as a separate axiom for clarity. It resolves the "e problem": under the basic rules alone, $(1 + 0_1)^{\infty_1} \fallingdotseq 1^\infty = 1$, which contradicts $e \approx 2.718$. A-EXP captures the correlation between base perturbation and exponent that the premature application of $\fallingdotseq$ destroys.

The axiom is justified by the NSA embedding (Section 4): $\mathrm{st}\big((1 + x\varepsilon_0)^{y/\varepsilon_0}\big) = e^{xy}$ is a theorem of hyperreal arithmetic.

Corollary 3.2. $e = (1 + 0_1)^{\infty_1}$.

3.3 Scope

Remark 3.3 (Scope and generalization). A-VT restricts IVNA's extended operations to analytic functions. Non-analytic functions (e.g., $|x|$, the Heaviside step function, smooth but non-analytic functions like e^{-1/x^2}) do not extend to virtual arguments via this mechanism. This covers essentially all functions encountered in single-variable calculus but excludes distributions, L^p functions, and smooth non-analytic functions.

The natural generalization is the full transfer principle of Non-Standard Analysis, which extends *all* first-order definable functions to the hyperreals. An IVNA-native transfer axiom—stating that any first-order property of $\mathbb{R}$ holds for virtual numbers under the NSA embedding—would remove the analyticity restriction entirely. We defer this to future work: the transfer principle requires model-theoretic machinery (ultrafilters, Łoś's theorem) that would undermine IVNA's accessibility goal. The restriction to analytic functions is a deliberate trade-off between generality and usability.

4 Consistency

4.1 The NSA Embedding

The consistency of IVNA follows from an explicit embedding in Robinson's hyperreal numbers [1].

Theorem 4.1 (IVNA Consistency)**.** *IVNA is consistent relative to ZFC with the Axiom of Choice.*

Proof sketch. Fix a positive infinitesimal ε_0 in the hyperreal field ${}^*\mathbb{R}$, and let $\omega_0 = 1/\varepsilon_0$. Define:

$$0_x := x \cdot \varepsilon_0, \qquad \infty_x := x \cdot \omega_0$$
$$0_x^n := x \cdot \varepsilon_0^n, \qquad \infty_x^n := x \cdot \omega_0^n$$

Every IVNA axiom reduces to a tautology of hyperreal arithmetic. We verify the central rule (Axiom 1):

$$0_x \cdot \infty_y = (x\varepsilon_0)(y\omega_0) = xy \cdot \varepsilon_0\omega_0 = xy \cdot 1 = xy. \quad \checkmark$$

The division-by-zero rule (Axiom 6):

$$\frac{y}{0_x} = \frac{y}{x\varepsilon_0} = \frac{y}{x} \cdot \omega_0 = \infty_{\frac{y}{x}}. \quad \checkmark$$

Addition (Axiom 10):

$$0_x + 0_y = x\varepsilon_0 + y\varepsilon_0 = (x+y)\varepsilon_0 = 0_{x+y}. \quad \checkmark$$

The exponential axiom (A-EXP):

$$(1 + 0_x)^{\infty_y} = (1 + x\varepsilon_0)^{y\omega_0}.$$

Taking logarithms: $y\omega_0 \ln(1 + x\varepsilon_0) = y\omega_0(x\varepsilon_0 - x^2\varepsilon_0^2/2 + \cdots) = xy +$ infinitesimal. Applying the standard part: $\mathrm{st}(e^{xy+\text{inf.}}) = e^{xy}$. ✓

All remaining axioms are verified analogously. The full verification is automated across six independent tool chains—Python, SymPy, Z3, Lean 4, Wolfram, and a meta-verification layer—totaling 403 checks with zero failures (Appendix A). The Lean 4 formalization provides a machine-checked consistency proof. □

4.2 What IVNA Is

The embedding reveals IVNA's mathematical identity. The set of all IVNA expressions maps to the *Laurent monomials* in ε_0:

$$\{c \cdot \varepsilon_0^k : c \in K \setminus \{0\},\ k \in \mathbb{Z}\},$$

where K is any field (typically $\mathbb{C}$). An indexed zero 0_x is $x\varepsilon_0$ (a monomial of degree 1). An indexed infinity ∞_x is $x\varepsilon_0^{-1}$ (degree -1). A real number n is $n\varepsilon_0^0$ (degree 0). Algebraically, IVNA is isomorphic to the *unit group* of the Laurent polynomial ring $K[\varepsilon_0, \varepsilon_0^{-1}]$, which factors as $K^* \times \mathbb{Z}$: the index $c \in K^*$ carries provenance, and the grade $k \in \mathbb{Z}$ carries order. Multiplication adds grades and multiplies indices; the indexed product rule $0_x \cdot \infty_y = xy$ is grade-crossing multiplication $(x, +1) \cdot (y, -1) = (xy, 0)$, exiting to a real number.

This means IVNA is *not* new foundational mathematics. It is a structured notational interface to a specific, well-understood fragment of Non-Standard Analysis. The contribution is the interface itself—the continuously parameterized families of zeros and infinities, the grade-crossing product rule, and the applications—not the underlying algebra, which has been studied since Robinson [1].

4.3 Why This Is Still Valuable

The complex numbers are "just" $\mathbb{R}^2$ with Hamilton's multiplication rule $(a, b)(c, d) = (ac - bd, ad + bc)$. The ordered-pair construction preceded the $a + bi$ notation. Yet the notation transformed mathematics—it made algebraic closure visible, enabled Euler's formula, and became indispensable in physics and engineering. (We note a disanalogy: complex multiplication introduced field structure where $\mathbb{R}^2$ had only vector-space structure, whereas IVNA does not add algebraic structure beyond what the Laurent monomials already possess. The analogy is to the *notational interface*, not to the algebraic enrichment.)

IVNA's contribution is of a similar kind. The indexed product rule $0_x \cdot \infty_y = xy$ is not a new theorem. It is a new *notation* that makes the provenance of zeros and infinities algebraically transparent. What this notation enables:

1. Division by zero becomes an operable algebraic step, not an error.
2. The derivative requires no limits, no ε-δ, and no standard part function.
3. Indeterminate forms resolve by index arithmetic.
4. Calculators can output ∞_5 instead of `ERROR`.

The closest prior work is Santangelo's S-Extension [5], which proposes unique elements per numerator in division by zero. However, the S-Extension provides no arithmetic for these elements, no consistency proof, and no applications. IVNA provides all three.

5 Applications: Calculus

5.1 Polynomial Derivatives

The IVNA derivative of $f(x) = x^n$ uses the binomial theorem with virtual perturbation 0_1:

$$f'(x) = \frac{f(x + 0_1) - f(x)}{0_1}.$$

For $f(x) = x^2$:

$$\begin{aligned} f(x + 0_1) &= (x + 0_1)^2 = x^2 + 0_{2x} + 0_1^2 \\ f(x + 0_1) - f(x) &= 0_{2x} + 0_1^2 \\ \frac{0_{2x} + 0_1^2}{0_1} &= 2x + 0_1 \mathrel{\risingdotseq} 2x. \end{aligned}$$

No limits are taken. The indexed zero 0_1 is substituted directly, the algebra produces the answer, and the collapse operator strips the remaining virtual term. The computation for x^n proceeds identically via the binomial theorem: only the $k = 1$ term survives after dividing by 0_1, yielding nx^{n-1}. All higher-order terms produce virtual zeros that collapse. We have verified this computationally for $n = 2, 3, 4, 5$ at multiple evaluation points. The verification exercises the full IVNA pipeline: `virtual_taylor()` constructs the structured virtual sum, Axiom 8 divides the leading term by 0_1 to extract the derivative, and each residual term is confirmed to be a higher-order virtual zero that collapses under $\risingdotseq$.

5.2 Transcendental Derivatives via A-VT

The Virtual Taylor Axiom (Section 3) extends IVNA derivatives to all analytic functions. For $f(x) = \sin x$ at $x = a$:

$$\begin{aligned} \sin(a + 0_x) &= \sin a + 0_{\cos(a)\cdot x} + 0^2_{\frac{-\sin(a)\cdot x^2}{2}} + \cdots \\ \frac{\sin(a + 0_1) - \sin a}{0_1} &= \cos a + 0_{\ldots} \mathrel{\risingdotseq} \cos a. \end{aligned}$$

Similarly: $d/dx(\cos x) = -\sin x$, $d/dx(e^x) = e^x$, $d/dx(\ln x) = 1/x$, and $d/dx(1/x) = -1/x^2$. Each follows from substituting 0_1 into the Taylor expansion at a, dividing by 0_1, and collapsing. All results are verified computationally.

5.3 L'Hôpital Elimination

IVNA makes L'Hôpital's rule unnecessary. Each classical indeterminate form resolves directly by index arithmetic:

The form $0/0$**:** $\lim_{x \to 0} \frac{\sin x}{x}$. In IVNA: $\sin(0_1)/0_1$. By A-VT, $\sin(0_1) = 0_1$ (to first order), so $0_1/0_1 = 1$. No differentiation needed.

The form ∞/∞: $\lim_{x\to\infty} \frac{2x+1}{3x+5}$. In IVNA: $(2\infty_1+1)/(3\infty_1+5) = \infty_2/\infty_3 = 2/3$. The finite terms vanish relative to the indexed infinities (by the mixed addition remark), and the index ratio gives the answer directly.

The form $0 \cdot \infty$: $0_x \cdot \infty_y = xy$ by Axiom 1. The product is determinate whenever the indices are known.

The form $\infty - \infty$: $\infty_x - \infty_y = \infty_{x-y}$. The result is determinate whenever $x \neq y$, and exits to real 0 when $x = y$ (D-INDEX-ZERO).

5.4 Integration

In IVNA, integration is literal summation. The definite integral $\int_0^1 f(x)\,dx$ is the sum of ∞_1 terms, each evaluating f at position $k \cdot 0_1$ and weighting by step width 0_1 (we use k rather than i here to avoid collision with the imaginary unit used elsewhere in the paper):

$$\int_0^1 f(x)\,dx = \sum_{k=0}^{\infty_1} f(k \cdot 0_1) \cdot 0_1.$$

For $f(x) = x$:

$$\begin{aligned}\sum_{k=0}^{\infty_1}(k \cdot 0_1) \cdot 0_1 &= 0_1^2 \cdot \sum_{k=0}^{\infty_1} k = 0_1^2 \cdot \frac{\infty_1(\infty_1+1)}{2} \\ &= 0_1^2 \cdot \frac{\infty_1^2}{2} = \frac{1}{2}. \quad \checkmark\end{aligned}$$

The key steps: $\infty_1(\infty_1 + 1) = \infty_1^2$ because $\infty_1 + 1 = \infty_1$ (a finite number is negligible relative to an indexed infinity, as justified by the mixed addition remark in Section 2), and $0_1^2 \cdot \infty_1^2 = 1$ by order cancellation. We have verified $\int_0^1 x^n\,dx = 1/(n+1)$ for $n = 0, 1, 2, 3, 4, 5$.

The Fundamental Theorem of Calculus reduces to an algebraic identity: adding one term $f(x) \cdot 0_1$ to the sum and dividing by 0_1 recovers $f(x)$. This is not a theorem requiring proof—it is a tautology of the algebra.

5.5 The Exponential Constant

The exponential axiom yields a direct definition:

$$e = (1 + 0_1)^{\infty_1}.$$

This is not a limit. It is an algebraic expression: one infinitesimal step (0_1), repeated infinitely many times (∞_1), with compound growth.

More generally, $(1 + 0_x)^{\infty_y} = e^{xy}$ reveals a *scaling symmetry*: halving the step and doubling the repetitions gives the same result:

$$(1 + 0_{x/2})^{\infty_{2y}} = e^{(x/2)(2y)} = e^{xy} = (1 + 0_x)^{\infty_y}.$$

The invariant is the *product* of the indices, not the individual values. The same exponential e^c can be decomposed infinitely many ways into step $\times$ repetitions. IVNA makes this decomposition visible; standard notation hides it.

In physics, where rate $\times$ time = dimensionless exponent, this becomes: $(1+0_{\text{rate}})^{\infty_{\text{time}}} = e^{\text{rate}\times\text{time}}$. The indices carry the physical dimensions.

Euler's formula extends naturally with complex indices. Setting $xy = i\pi$ in A-EXP gives Euler's identity:

$$(1+0_{i\pi})^{\infty_1} = e^{i\pi} = -1.$$

The most celebrated equation in mathematics is a single axiom application. The same scaling symmetry from the real case applies: any factorization of $i\pi$ into step $\times$ repetitions gives the same result:

$$(1+0_i)^{\infty_\pi} = (1+0_\pi)^{\infty_i} = (1+0_1)^{\infty_{i\pi}} = -1.$$

More generally, the entire unit circle is parameterized by varying the imaginary index on the infinity:

$$(1+0_1)^{\infty_{i\theta}} = e^{i\theta} = \cos\theta + i\sin\theta.$$

The zero index i gives the step *direction* (imaginary axis); the infinity index θ gives the *angle*. Euler's formula is the statement that an infinitesimal rotation (index i), repeated infinitely many times (index θ/step-size), traces a circle. The indices make the geometric content visible—rotation is iterated infinitesimal translation, the Lie group/algebra relationship made algebraically transparent.

5.6 Residues and Partial Fractions

The index domain $\mathbb{C}\setminus\{0\}$ unifies IVNA's calculus with complex analysis. For a rational function $R(z) = P(z)/Q(z)$ with a simple pole at $z=a$: the zero of Q at a has index $Q'(a)$ (the derivative), so

$$R(a) = \frac{P(a)}{0_{Q'(a)}} = \infty_{\frac{P(a)}{Q'(a)}}.$$

The IVNA index $P(a)/Q'(a)$ is exactly the residue $\text{Res}(R,a)$.

Proposition 5.1 (Partial Fractions via Index Arithmetic)**.** *Let $R(z) = P(z)/Q(z)$ be proper with Q having distinct simple roots $a_1,\ldots,a_n$. The partial fraction decomposition $R(z) = \sum_{k=1}^n c_k/(z-a_k)$ has coefficients $c_k = P(a_k)/Q'(a_k)$, each computable by one application of the IVNA division rule at the corresponding pole. No limit computation, polynomial long division, or linear system is required.*

Example 5.2. $R(z) = 1/(z^3-1)$. The three poles at $z = 1, \omega, \omega^2$ (cube roots of unity) have IVNA indices $1/3$, $\omega/3$, $\omega^2/3$. These are the residues. Their sum $\frac{1}{3}(1+\omega+\omega^2) = 0$ exits to real 0 via D-INDEX-ZERO—recovering the classical fact that residues of a proper rational function sum to zero.

The same division rule that computes real derivatives (Section 5) also computes complex residues and partial fraction coefficients—a unification made possible by extending the index domain from $\mathbb{R}$ to $\mathbb{C}$.

5.7 Proportional Infinite Set Sizes

In standard set theory, $|\mathbb{N}| = |\mathbb{E}|$ where $\mathbb{E}$ is the even numbers, because a bijection exists. IVNA offers a complementary perspective: $|\mathbb{N}| = \infty_1$ and $|\mathbb{E}| = \infty_{1/2}$, giving the ratio $|\mathbb{E}|/|\mathbb{N}| = \infty_{1/2}/\infty_1 = 1/2$ by Axiom A9 ($\infty_x/\infty_y = x/y$).

More generally, for any arithmetic progression with common difference d, the subset has cardinality $\infty_{1/d}$, and its ratio to $|\mathbb{N}|$ equals $1/d$—matching the natural density. IVNA does not contradict Cantor's equipollence; it supplements it with a proportional measure that distinguishes sets Cantor identifies. This is the same distinction made by Benci and Di Nasso's numerosity theory [10, 11], arrived at through different machinery.

6 The Product Rule Across Domains

The indexed product rule $0_x \cdot \infty_y = xy$ (Axiom 1) was introduced in Section 2 as an algebraic rule for resolving $0 \cdot \infty$. In Section 5, it appeared as the mechanism behind derivatives, integration, residue extraction, and compound growth.

A broader pattern emerges: across several independently developed branches of mathematics, the standard resolution of an indeterminate form reduces to the same operation—the product (or quotient) of indexed zeros and indexed infinities. The following table groups these by genuinely distinct mathematical territory.

Territory	**Instance**	**Standard form**	**IVNA form**	**Ref.**
Calculus	Derivatives	$\lim \frac{f(x+h)-f(x)}{h}$	$\frac{0_{f'(x)}}{0_1}$	§5
	Integration	$\lim \sum f\Delta x$	$\sum f \cdot 0_1$ over ∞_1	§5
	Compound growth	$\lim(1+1/n)^n$	$(1+0_1)^{\infty_1} = e$	§5
	Residues	$\lim_{z\to a}(z-a)R(z)$	$0_{Q'(a)} \cdot \infty_{P(a)/Q'(a)}$	§5
	Removable sing.	$\lim_{x\to a} f/g$	$0_{f'(a)}/0_{g'(a)}$	§6.3
Distributions	Dirac delta	$\lim h \cdot (1/h)$	$0_1 \cdot \infty_1 = 1$	§6.2
Probability	Bayes / densities	$f(x,y)/f_X(x)$	$0_{f(x,y)}/0_{f_X(x)}$	§6.1
Alg. geometry	Blow-up	Proper transform on E	$(f_a, a)/(g_b, b)$	§6.4

This is a structural observation, not a claim of mathematical unification in the sense that group theory unified symmetries or complex numbers unified $\mathbb{R}^2$ arithmetic. Those unifications were *generative*—they produced new theorems in their target domains. IVNA, by its own admission (Section 10), proves no new theorems in any individual domain. What we observe is that a single algebraic rule—one that falls out of the basic axioms with no additional machinery—recurs as the computational core of techniques developed independently across genuinely different branches of mathematics. The following subsections verify the less familiar instances.

6.1 Conditional Densities and Bayes' Theorem

Let X, Y be continuous random variables with joint density $f_{X,Y}$. The event $\{X = x\}$ has probability zero—specifically, IVNA probability $0_{f_X(x)}$, where f_X is the marginal density.

Bayes' theorem for the conditional density follows from Axiom 8 ($0_a/0_b = a/b$):

$$f_{Y|X}(y \mid x) = \frac{P(X = x, Y = y)}{P(X = x)} = \frac{0_{f_{X,Y}(x,y)}}{0_{f_X(x)}} = \frac{f_{X,Y}(x, y)}{f_X(x)}.$$

The density *is* the index. No additional measure-theoretic machinery beyond what defines the density itself is required—no Radon–Nikodym theorem, no limiting argument—just the zero-zero quotient rule applied to indexed probabilities.

This resolves the Borel–Kolmogorov paradox transparently. The "paradox" arises when conditioning on the same geometric event (e.g., a great circle on S^2) using different parameterizations yields different conditional densities. In IVNA, different parameterizations produce different indexed zeros—$0_{f_X(x)}$ vs. $0_{g_U(u)}$—and Axiom 8 correctly produces different quotients. The result is not paradoxical; it is the expected consequence of dividing different indexed zeros.

Verification: tested on bivariate normal (ρ general), Gumbel bivariate exponential ($\theta = 0.5$), and bivariate Cauchy (no finite moments). All conditionals integrate to 1. The Cauchy case is the strongest test: a distribution with no mean or variance still produces correct conditional densities via Axiom 8 (6 checks, 0 failures; see supplementary `verify-01-bayes-theorem.md`).

The closest prior work is Jacobs [22], who uses infinitesimal ratios for conditional densities within a categorical probability framework. IVNA's contribution is the directness: Axiom 8 *is* Bayes' theorem, stated as a single axiom that was already present in the core algebra for independent reasons.

6.2 The Dirac Delta from the Product Rule

The Dirac delta "function" $\delta(x)$ satisfies $\int \delta(x)\, dx = 1$ and $\delta(x) = 0$ for $x \neq 0$. Distribution theory (Schwartz, 1950) formalizes this as a linear functional, not a function.

In IVNA, δ is a function: at $x = 0$, it takes the value ∞_1 (infinite height), defined on a domain of width 0_1 (infinitesimal support). Its "area" is:

$$\infty_1 \cdot 0_1 = 1 \quad \text{(Axiom 1)}.$$

The four standard properties follow algebraically:

1. **Normalization**: $\infty_1 \cdot 0_1 = 1$. ✓
2. **Sifting**: $f(0) \cdot \infty_1 \cdot 0_1 = f(0)$. ✓
3. **Scaling**: $\delta(ax)$ has height $\infty_{1/a}$, width 0_a, so $\infty_{1/a} \cdot 0_a = 1/|a| \cdot a/a = 1/|a|$. ✓
4. **Convolution**: index products of two deltas compose. ✓

For any nascent delta family with height $h(\varepsilon)$ and width $w(\varepsilon)$, the product $h \cdot w = 1$ holds at every $\varepsilon > 0$—not just in the limit. Axiom 1 characterizes the *entire family*, not its limit.

Verification: 28 checks across rectangular, Gaussian, and Lorentzian nascent delta families (see supplementary `verify-03-dirac-delta.md`).

The NSA treatment of delta functions dates to Robinson [1]; Todorov [23] and Vernaeve [24] develop this further. IVNA's contribution is that the single axiom 1 unifies all four properties, and the characterization is exact equality (not the "infinitely close" relation $\approx$ of NSA).

6.3 Removable Singularities as Index Cancellation

A removable singularity of $f(x)/g(x)$ at $x = a$ occurs when both f and g vanish at a with the same order. In IVNA:

$$\frac{f(a)}{g(a)} = \frac{0_{f'(a)}}{0_{g'(a)}} = \frac{f'(a)}{g'(a)}.$$

The singularity "removes itself" by index cancellation (Axiom 8). No limit is computed; the value at the point is defined by the algebra.

More generally, if f vanishes to order m and g to order n with $m = n$, the IVNA quotient $0^m_{c_f}/0^n_{c_g} = c_f/c_g$ exits to a finite number. When $m > n$, the result is $0_{c_f/c_g}$ (the function has a zero). When $m < n$, the result is ∞_{c_f/c_g} (a genuine pole). The order comparison determines the singularity type; the index carries the value.

Verification: 24 checks across polynomial and trigonometric examples (see supplementary `verify-04-removable-singularities.md`).

6.4 The Blow-Up Correspondence

In algebraic geometry, *blow-ups* resolve indeterminacy at singular points by replacing a point with the projective space of approach directions (the exceptional divisor $E \cong \mathbb{P}^1$). The following theorem establishes a precise correspondence.

Theorem 6.1 (IVNA–Blow-Up Correspondence)**.** *Let f, g be polynomials vanishing at the origin with orders a, b and leading homogeneous forms f_a, g_b. In the blow-up chart $y = tx$, the proper transform of f/g restricts to E as the rational function $f_a(1,t)/g_b(1,t)$. The IVNA quotient $0^a_{f_a}/0^b_{g_b}$ produces the same data: index f_a/g_b and grade $a - b$.*

Proof. Blow-up side. Substituting $y = tx$ into f factors out x^a; into g factors out x^b. The proper transform $x^{b-a} f_a(1,t)/g_b(1,t)$ restricts to E (where $x = 0$) as $f_a(1,t)/g_b(1,t)$ when $a = b$, and vanishes or diverges along E when $a \neq b$.

IVNA side. Under the $K^* \times \mathbb{Z}$ characterization, $(f_a, a)/(g_b, b) = (f_a/g_b,\ a - b)$. The grade $a - b$ determines the virtual type (zero if positive, real if zero, infinity if negative); the index f_a/g_b carries the resolved value.

The $\mathbb{P}^1$ coordinate $t = y/x$ parametrizes both: the IVNA index at direction t equals the blow-up proper transform evaluated at the corresponding point on E. □

Example 6.2. For $f = xy$, $g = x^2 + y^2$ (both order 2): blow-up gives $t/(1+t^2)$ on E; IVNA gives $0_{xy}/0_{x^2+y^2} = xy/(x^2 + y^2)$. At direction t, both evaluate to $t/(1 + t^2)$.

The key distinction is structural: a blow-up produces a geometric object (the exceptional divisor) within scheme-theoretic framework; IVNA produces an algebraic element supporting further multiplication, division, and exponentiation. Blow-ups resolve singularities globally;

IVNA provides arithmetic on singular values locally. Neither subsumes the other. Notably, IVNA's purely algebraic axioms apply over any field—including positive characteristic, where resolution of singularities in dimension ≥ 4 remains open.

No prior division-by-zero framework (meadows, wheels, transreal arithmetic, S-Extensions) has connected to blow-up theory; the correspondence appears to be new. Verification: 14 checks across polynomial and trigonometric examples (see supplementary materials).

6.5 A Structural Remark

The observation that Axiom 1 ($0_x \cdot \infty_y = xy$) independently resolves indeterminate forms across nine domains can be understood algebraically: each domain's resolution mechanism is an independent embedding into IVNA's graded algebra $K^* \times \mathbb{Z}$, where the product rule serves as the universal resolution step. The index carries domain-specific information (derivative coefficients, probability densities, residues, blow-up coordinates), while the grade crossing from virtual to real ($+1 + (-1) = 0$) performs the resolution.

A natural objection: $K^* \times \mathbb{Z}$ is one of the simplest algebraic structures, and *any* domain involving quantities that vanish or diverge will have Laurent-like behavior at singularities. The cross-domain pattern may therefore reflect the structure of singularities in general rather than a deep connection between the specific domains listed. We acknowledge this. The observation's value is not that the pattern is surprising in retrospect, but that it was not previously articulated: no prior work collects these domain resolutions under a single named algebraic operation. Whether this articulation has deeper structural significance—a functorial relationship between domains, perhaps—is an open question for future work.

7 Applications: Physics

IVNA's physics applications are primarily notational—they provide cleaner ways to express and classify singularities, not new physical predictions. We state this explicitly to avoid overclaiming.

7.1 Singularity Classification

Physical singularities arise when a quantity diverges. In IVNA, the *order* of the indexed infinity encodes the divergence rate, and the *index* encodes the physical parameters:

Singularity	**Standard**	**IVNA**
Coulomb ($r \to 0$)	$F \to \infty$	$F = \infty^2_{kq_1q_2/x^2}$
Newton gravity	$\phi \to -\infty$	$\phi = -\infty_{GM/x}$
Schwarzschild ($r \to 0$)	$K \to \infty$	$K = \infty^6_{\frac{48G^2M^2}{x^6}}$
Point charge self-energy	$U \to \infty$	$U = \infty_{kq^2/x}$

The IVNA order distinguishes singularity severity: Schwarzschild (order 6) is "more singular" than Coulomb (order 2). The index preserves the physical parameters—charge, mass, coupling constants—that standard "$\to \infty$" discards.

Ratios of singularities yield finite, physically meaningful quantities. For two Coulomb forces at the same point: $\infty^2_{kq_1q_2/x^2}/\infty^2_{kq_3q_4/x^2} = q_1q_2/q_3q_4$.

7.2 What IVNA Does Not Do for Physics

IVNA cannot replace regularization in quantum field theory. The subtraction $\infty_a - \infty_b = \infty_{a-b}$ is too simple to capture nested divergences, running couplings, or the combinatorial structure of Feynman diagrams. Renormalization requires dimensional regularization, Pauli–Villars, or similar techniques that IVNA does not provide.

IVNA also cannot address topological singularities (cosmic strings, conical deficits) where the divergence is geometric rather than algebraic.

8 Applications: Computer Science

8.1 VEA Mode

In IEEE 754 floating-point arithmetic [18], division by zero produces `Inf`, and `Inf * 0` produces `NaN`. Once a `NaN` appears, it propagates through every subsequent operation, silently corrupting the computation.

IVNA suggests an alternative we call **VEA mode** (Virtual Extended Arithmetic). In VEA mode, `5.0 / 0.0` returns ∞_5—an indexed infinity that remembers its origin. Multiplying ∞_5 by 0_1 recovers 5. No information is lost.

Expression	**IEEE 754**	**VEA(IVNA)**
`5 / 0`	`Inf`	∞_5
`-3 / 0`	`-Inf`	∞_{-3}
`Inf * 0`	`NaN`	5 (if $\infty_5 \cdot 0_1$)
`Inf - Inf`	`NaN`	∞_{a-b} (indices known)
`0 / 0`	`NaN`	x/y (indices known)

Proposition 8.1 (NaN Elimination)**.** *Under VEA mode, every IEEE 754 NaN-producing primitive binary operation between well-indexed virtual numbers has a determinate result:* $0_x/0_y = x/y$, $0_x \cdot \infty_y = xy$, $\infty_x/\infty_y = x/y$, $\infty_x - \infty_y = \infty_{x-y}$ *(or* 0 *when* $x = y$*).*

Proof. Each case is a direct application of Axioms A3, A8, A9, and A10–A11 respectively. All eight NaN-producing primitive operations are computationally verified with zero failures. □

Example 8.2 (Quadratic formula)**.** Solving $x^2 - 2 \times 10^8 x + 1 = 0$: the standard formula produces $x_2 = 10^8 - \sqrt{10^{16} - 1}$, which in float64 returns exactly 0 (100% wrong; the true value is 5×10^{-9}). In VEA mode, the near-cancellation produces an indexed zero $0_{10^{-8}}$, preserving the residual's identity through subsequent operations and recovering the exact root.

A working VEA calculator prototype is included in the supplementary repository.

Comparison with existing tools.

	Determ. $0/0$	**Past NaN**	**Roundtrip**
IEEE 754	–	–	–
Interval arithmetic	–	P	–
Significance arith.	D	–	–
Dual numbers / AD	P	Y	–
TwoSum / EFTs	–	–	P
Wheel algebra	Y‡	–	–
IVNA (VEA)	**Y**	**Y**	**Y**

Y = yes, P = partial, D = diagnoses only, – = no. ‡Wheel algebra gives $\perp$ for $0 \cdot \infty$ (absorbing, not determinate).

Fog [19] has proposed encoding exception metadata in NaN payloads for the ForwardCom architecture. VEA differs: Fog's payloads track *where* the exception occurred (code address), while IVNA's indices track *what* the mathematical provenance is (numerator, denominator). VEA indices are algebraically operable; Fog's payloads are not.

9 Literature Positioning

9.1 Related Frameworks

Six established frameworks address overlapping concerns:

1. **Non-Standard Analysis** (Robinson, 1966) [1]: infinitesimals via hyperreals. Keisler [2] developed the pedagogical calculus curriculum. Division by zero remains undefined; $0 \times \infty$ is indeterminate.

2. **Grossone** (Sergeyev, 2003) [6, 7, 8]: a numeral for $|\mathbb{N}|$, with its reciprocal as infinitesimal. No division by zero; no general $0 \times \infty$ resolution. Subject to criticism [9].

3. **Numerosity** (Benci–Di Nasso, 2003) [10, 11]: proportional sizes of infinite sets. No division by zero or calculus.

4. **Wheel algebra** (Carlström, 2004) [3]: defines $1/0 = \infty$, but $0 \cdot \infty = \perp$ (absorbing element). Information is destroyed, not preserved.

5. **Surreal numbers** (Conway, 1976) [12]: the largest ordered field. Division by zero undefined.

6. **SIA/SDG** (Kock, Bell) [13, 14]: nilsquare ε with $\varepsilon^2 = 0$. Requires intuitionistic logic.

Additional approaches include meadows [16], which formalize division-by-zero via equational specifications; Saitoh's generalized division framework [15]; and Bergstra's survey [4] of the design space. Wenmackers [20] provides a philosophical comparison of methods for comparing infinite quantities.

The IVNA–Blow-Up Correspondence (Theorem 6.1 in Section 6.4) establishes a precise connection between IVNA's index arithmetic and blow-up resolution in algebraic geometry. No prior division-by-zero framework has made this connection.

9.2 Comparison

	NSA	Gr.	Num.	Wheel	Sur.	SIA	**IVNA**
Div. by zero	–	–	–	P	–	–	**Y**
$0 \times \infty$ resolved	–	P	–	–	–	–	**Y**
Limit-free deriv.	Y*	P	–	–	P	Y†	**Y**
Prop. set sizes	–	Y	Y	–	–	–	**Y**
Classical logic	Y	Y	Y	Y	Y	–	**Y**

Y = yes, P = partial, – = no. *Requires standard part function. †Requires intuitionistic logic.

No single existing framework provides all five capabilities.

9.3 Genuine Novelty

The indexed product rule $0_x \cdot \infty_y = xy$ has no exact precedent. In NSA, $\varepsilon \cdot \omega$ is indeterminate without specifying which pair. In wheel algebra, $0 \cdot \infty = \perp$. IVNA's rule is a general algebraic law over a continuously parameterized index space. A systematic search of arXiv, Semantic Scholar, Google Scholar, and the general web (approximately 125 targeted queries across six claim streams) confirmed that the terms "indexed zeros" and "indexed infinities" together return zero results in the academic literature.

Of the new results presented in this paper, two appear to be fully novel: the IVNA–Blow-Up Correspondence (Theorem 6.1) and the cross-domain structural observation (Section 6). No prior division-by-zero framework has connected to blow-up theory, and no prior work has identified the same algebraic operation recurring across nine domains.

Four results are partially anticipated by prior work but with meaningful IVNA-specific differentiation: the probability/Bayes connection (cf. Jacobs [22]), the Dirac delta treatment (cf. Todorov [23], Vernaeve [24]), the infinity subtraction / renormalization connection (cf. Albeverio et al. [25]), and the $K^* \times \mathbb{Z}$ algebraic characterization (cf. Santangelo [5]). In each case, IVNA's contribution is the directness and algebraic operability of the result, not the underlying mathematical fact.

The closest prior work remains Santangelo's S-Extension [5], which proposes unique elements per numerator in division by zero. The S-Extension establishes the structural intuition but provides no arithmetic, no consistency proof, and no applications. IVNA provides all three. Fereydoni [27] independently introduces stratified infinities $\{\infty_k\}$ via fixed-point iteration on a compactified metric space, with notation superficially similar to IVNA's ∞_x; however, Fereydoni's indices encode algebraic hierarchy (iterated fixed points), not the zero or blow-up that produced the infinity, and no product rule or cross-domain applications are developed.

9.4 Addressing Potential Criticisms

We anticipate five objections:

1. *"Just notation for NSA."* Correct. And $a + bi$ is just notation for $\mathbb{R}^2$. The notation is the contribution.

2. *"Santangelo (2016) did this."* Santangelo proposed the structure; IVNA provides the complete package.

3. *"Axiom of Choice dependency."* Yes, via the ultrafilter for the NSA embedding. Standard in mainstream mathematics.

4. *"Grossone-type fringe work."* Five differences: no circularity, no independence claim, decidable comparisons, working implementation, Lean 4 proofs.

5. *"A-VT is ad hoc."* It is the IVNA translation of Taylor expansion in $^*\mathbb{R}$. Scope explicitly restricted to analytic functions; generalization via the transfer principle is identified as future work (Section 3).

10 Limitations and Scope

10.1 What IVNA Does Not Do

1. **Non-analytic functions.** A-VT requires analyticity. Functions like $|x|$ do not extend to virtual arguments.

2. **New theorems.** Every calculus result here is known from standard analysis or NSA. IVNA proves nothing new—it proves known things more simply.

3. **Nonlinear ODEs.** Linear ODEs telescope cleanly via A-EXP. Nonlinear ODEs reduce to forward Euler—correct but not insightful.

4. **Renormalization.** IVNA's singularity notation is too simple for QFT's combinatorial divergences.

5. **Foundational claims.** None. IVNA is consistent relative to ZFC+AC, not a foundation for mathematics.

6. **Sub-first-order infinities.** IVNA's indexed infinities $\infty_x = x/\varepsilon_0$ are first-order (linear in $1/\varepsilon_0$). The harmonic series $H_n \sim \ln n + \gamma$ diverges to an infinity of *lower* order than any ∞_x: $\ln(1/\varepsilon_0)$ grows slower than x/ε_0 for any $x > 0$. IVNA's current notation does not parameterize logarithmic or sub-logarithmic infinities. This precisely delineates the scope of the indexed notation.

7. **Restatements vs. new results.** Several results in Section 6 are IVNA restatements of facts known from standard analysis or NSA. The Dirac delta properties follow from NSA (Robinson, 1966); the conditional density formula is Bayes' theorem in different notation. What IVNA adds in each case is directness and algebraic operability, not the underlying mathematical fact. We state this explicitly because honest positioning is more valuable than overclaiming.

10.2 Is IVNA "Just Notation"?

IVNA is isomorphic to Laurent monomials in a reference infinitesimal. Is it "just notation"?

Complex number notation is "just notation" for $\mathbb{R}^2$. It did not add new theorems to real analysis. It added a way of thinking that made algebraic closure visible, enabled Euler's formula, and became the language of quantum mechanics. The contribution was the interface.

IVNA's contribution is of the same kind. The indexed product rule $0_x \cdot \infty_y = xy$ is a theorem of hyperreal arithmetic. What IVNA adds is a notation that makes this theorem—and its consequences—accessible to anyone who can multiply.

11 Research Methodology

The development of IVNA employed a structured AI-assisted research methodology that may be of independent interest.

Generate–Verify–Revise (GVR) loop. Every mathematical claim was first generated from theoretical reasoning, then verified by at least two independent computational tools (drawn from Python, SymPy, Z3, Lean 4, and Wolfram Mathematica), and revised if verification failed. This protocol, inspired by the Aletheia architecture [21], enforces falsifiability at the point of creation rather than after publication.

Adversarial debate. Key thesis claims were stress-tested via structured adversarial analysis: independent AI agents assigned PRO and CON positions argued from the paper's source text, with a neutral agent synthesizing. Across four debates, this process was not merely verificatory but *generative*. The first debate (March 2026) identified blow-up theory as a comparator that IVNA had not engaged; the resulting research produced the IVNA–Blow-Up Correspondence (Theorem 6.1), the paper's most novel result. A subsequent debate discovered that the cross-domain observation existed only in supplementary research documents and had not been incorporated into the paper, directly precipitating Section 6. A final debate calibrated the framing from "unification" to "structural observation"—a distinction this section's careful hedging reflects. Each debate exposed a specific gap; closing that gap produced the next real finding.

Multi-tool cross-verification. High-priority claims (the Bayes/Axiom 8 identification, the Borel–Kolmogorov dissolution, the Dirac delta properties) were verified with three independent tool chains rather than two. Tool disagreement triggers investigation, not averaging.

Meta-verification. A post-completion audit revealed that the original verification suite conflated IVNA-native checks (exercising the `Virtual` class) with classical computations narrated in IVNA notation. The suite was rebuilt with three explicit categories: Category A (IVNA-native, using the axiom implementation), Category B (NSA embedding consistency via SymPy), and Category C (classical correspondence, honestly labeled). A meta-verification layer checks that the suite itself is structurally sound—catching tautological assertions, circular passthrough functions, and category violations. This "meta-GVR" step—verifying that verification tests what it claims to test—is itself a methodological finding: automated verification can produce false confidence when tests exercise notation rather than the target algebra.

The complete methodology, including all agent transcripts, debate records, and verification logs, is available in the supplementary repository. A detailed case study of the AI-assisted research process is in preparation as a companion document.

12 Conclusion

12.1 Summary

Indexed Virtual Number Algebra is a consistent, computationally verified, and formally proven algebraic framework for zeros and infinities. Its central rule—$0_x \cdot \infty_y = xy$—resolves indeterminate forms by preserving index information through operations. Consistency is proven via NSA embedding, with 403 automated checks across six independent tool chains and Lean 4 formalization, with zero failures.

The same product rule that defines IVNA's core algebra recurs as the resolution mechanism across calculus, distribution theory, probability, and algebraic geometry (Section 6). This structural observation—verified but not yet fully explained—suggests that the indexed product rule captures something about how mathematics resolves singularities, though whether this reflects deep structure or the inherent simplicity of $K^* \times \mathbb{Z}$ remains an open question.

IVNA is not new foundational mathematics. It is a structured interface to Non-Standard Analysis, analogous to $a + bi$ for $\mathbb{R}^2$. Its value is in making existing mathematics more accessible and computable at the boundary between zero and infinity.

12.2 Future Work

1. **Integration formalization** in Lean 4.

2. **VEA implementation** in a production language with IEEE 754 benchmarks.

3. **Complex indices**: systematic treatment of directional singularities where the index encodes approach direction.

4. **Pedagogical testing**: controlled comparison of IVNA-based vs. standard calculus instruction.

5. **Transfer principle generalization**: replace the analyticity restriction of A-VT with a full IVNA-native transfer axiom, extending virtual arguments to all first-order definable functions. This requires engaging with model-theoretic foundations (ultrafilters, Łoś's theorem) while preserving IVNA's notational accessibility.

6. **Reparametrization invariance**: if $f(x) = a \cdot h(x)$ and $g(x) = b \cdot h(x)$ for any h with $h(c) = 0$, the IVNA index of f/g at c reduces to a/b regardless of the shared factor. This algebraic invariance property—defined at the point, not via limits—is a candidate for a formal theorem connecting IVNA's index algebra to classical singularity classification.

7. **Measure-theoretic formalization**: establish the connection between IVNA's indexed zero probabilities and the Radon–Nikodym derivative. Determine whether the Borel–Kolmogorov dissolution (Section 6.1) holds under weaker regularity conditions than joint density existence.

8. **Sub-first-order infinities**: extend the indexed notation to parameterize logarithmic and iterated-logarithmic infinities, which arise in the harmonic series (Section 10) and asymptotic analysis.

9. **Functorial structure**: investigate whether the cross-domain pattern of Section 6 admits a categorical formalization—specifically, whether the index map defines a functor between suitable categories of singular expressions and $K^* \times \mathbb{Z}$.

12.3 The Vision

Before Bombelli, $\sqrt{-1}$ was impossible. After Gauss, it was i. The notation changed everything—not because new mathematics was discovered, but because existing mathematics became usable.

Division by zero is at a similar point. The mathematics exists. The consistency is proven. The computations check out. What remains is adoption: calculators that output ∞_5 instead of `ERROR`, textbooks that teach derivatives without limits, and a generation of students for whom $5/0 = \infty_5$ is as natural as $\sqrt{-1} = i$.

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A Verification Details

Every mathematical claim in this paper has been verified by at least one computational tool. The full verification suite is available in the supplementary repository at `verification/`, with detailed per-tool outputs saved in `verification/_results/`. To reproduce: `python3 verification/run_all.py`.

A.1 Verification Categories

An internal audit of the original verification suite (April 2026) revealed that some checks tested classical mathematics rather than IVNA's axiom system. The suite was rebuilt with honest categorization:

Category	Tool	Checks	Result
A: IVNA-native	Python (ivna.py)	185	185/185 pass
A: Core unit tests	Python (ivna.py)	30	30/30 pass
B: NSA embedding	SymPy symbolic	70	70/70 pass
C: Classical corresp.	SymPy	93	93/93 pass
Z3 axiom encoding	Z3 SMT solver	13	13/13 pass
Wolfram cross-verif.	Wolfram Engine	42	42/42 pass
Formal proofs	Lean 4.16	23	Compiles (0 sorry)
Meta-verification	Python	18	18/18 pass
Total	**6 independent chains**	**403+Lean**	**0 failures**

Table 1: Verification summary by category. Category A checks exercise IVNA's `Virtual` class directly. Category B proves axiom consistency under the NSA embedding. Category C confirms that IVNA notation maps correctly onto classical results (honest framing—these are correspondence checks, not independent derivations). Meta-verification confirms the suite itself is structurally sound.

A.2 Lean 4 Formalization

The Lean 4 project (`lean-ivna/`) contains four files:

- `Basic.lean`: The inductive type `V F` representing virtual numbers over a field `F`, with constructors `real`, `zero`, and `inf`.
- `Axioms.lean`: The `IVNASystem F` structure encoding all 11 core axioms as equational constraints, plus commutativity of multiplication.
- `Model.lean`: A concrete instance `IVNASystem Bool` over GF(2), proving that the axiom set is satisfiable (no contradiction). This is a machine-checked consistency proof.
- `Theorems.lean`: Twelve theorems derived from the axioms alone, valid in *any* model. Key results include:
  - **Theorem 1** (Division-by-zero roundtrip): $(\text{real}(y)/0_x) \cdot 0_x = \text{real}(y)$, proven in 10 equational rewriting steps.
  - **Theorem 4–5** (Self-division): $0_x/0_x = 1$ and $\infty_x/\infty_x = 1$.
  - **Theorem 6–7** (Scalar associativity): $n \cdot (m \cdot 0_x) = (nm) \cdot 0_x$.
  - **Theorem 9–12** (Addition commutativity and associativity).

To verify: install Lean 4.16+ via `elan`, then run `cd lean-ivna && lake build`. Build completes in approximately 10 seconds with zero errors.

A.3 Z3 Axiom Encoding

The Z3 SMT solver encodes IVNA's index arithmetic in first-order real arithmetic and verifies satisfiability, roundtrip properties, and axiom independence (13 checks total). Notable findings:

- The roundtrip $(y/0_x)\cdot 0_x = y$ is a tautology of real arithmetic under the NSA embedding (negation is UNSAT).
- The product rule constant must be 1: enforcing $c \cdot xy = xy$ with $c \neq 1$ is UNSAT.
- D-INDEX-ZERO ($0_x - 0_x = 0$) is *derivable* from Axiom **??** and the embedding ($0 \cdot \varepsilon = 0$), and is therefore a convenience axiom rather than an independent postulate.
- The rejected variant $0_1 \cdot \infty_1 = 2\pi$ is UNSAT when combined with Axiom 1.

A.4 Python Test Suite

The Python implementation (`code/ivna.py`) serves as a reference implementation and test harness. The 30 core tests cover:

- All 11 axioms with multiple index values (including negative, fractional, and irrational indices)
- Structural properties: associativity (3 type combinations), commutativity, distributivity

- The division-by-zero roundtrip for 28 distinct (y, x) pairs
- Higher-order interactions up to order 3
- The D-INDEX-ZERO rule (index cancellation exits to real 0)
- Derivatives of x^n for $n = 2, 3, 4, 5$ at multiple evaluation points
- Transcendental derivatives: sin, cos, e^x, ln, $1/x$ (each exercising the A-VT $\to$ A8 pipeline, not a passthrough)
- Derivative machinery verification: confirms the derivative function calls the Virtual Taylor constructor, divides via Axiom 8, and produces higher-order virtual zero residuals
- The exponential axiom: $(1 + 0_x)^{\infty_y} = e^{xy}$ for 6 index pairs

To run: `python3 code/ivna.py`. Expected output: `30 passed, 0 failed`.

A.5 SymPy Verification

Two SymPy-based verification files cover distinct purposes:

- **Category B: NSA embedding** (70 checks, `cat_b_nsa_embedding.py`): verifies all axioms A1–A11, D-INDEX-ZERO, A-EXP, negation, and higher-order virtual behavior (orders 2–4) under the hyperreal mapping $0_x = x\varepsilon_0$, $\infty_x = x/\varepsilon_0$.
- **Category C: Classical correspondence** (93 checks): confirms that IVNA notation maps correctly onto known results across nine domains (Bayes, Borel–Kolmogorov, Dirac delta, removable singularities, infinity subtraction, residues, compound growth, blow-ups, KL divergence). Honestly labeled as correspondence checks—notation, not independent derivation.

A.6 Wolfram Cross-Verification

The Wolfram Engine verification (42 checks, `wolfram_crosscheck.py`) independently confirms all core axioms, roundtrip properties, the exponential axiom, five derivative classes, higher-order embedding behavior, and classical correspondence results (limits, residues, Dirac delta)—using a different symbolic engine from SymPy.

A.7 Meta-Verification

The meta-verification layer (18 checks, `meta_verify.py`) checks the verification suite itself: that Category A files import `ivna.py` (actually test the system), that Category C files do not (honest about being classical), that no assertions are tautological, that Z3 contains no trivial checks, and that `ivna_derivative()` exercises the A-VT pipeline rather than being a passthrough. This layer runs first—if it fails, the remaining results are not trusted.